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Generalization and Modification of Hardy-Littlewood Maximal Functions

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ABSTRACT: The purpose of this paper is to provide the different types of Hardy-Littlewood Maximal Functions, the relationship between them and the corresponding extension of $\mathbb{R}^n$ of the Hardy-Littlewood maximal function. We also give the generalization and the modification of Hardy-Littlewood maximal function.

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Maximal functions arise very natural in analysis, for proving theorems about the existence almost everywhere of limits, for controlling pointwise important objects such as the Poisson Integrals or for controlling, not pointwise but at least in average, other basic operators such as singular integral operators. The model example of existence almost everywhere of limits is the Lebesgue differentiation theorem:

$$f(x) = \lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy$$

This property is intimately related to the study of certain properties of the Hardy-Littlewood maximal function. It is a classical mean operator, and it is frequently used to majorize other important operators in harmonic analysis. There are other almost everywhere convergence problems in mathematics: Fourier series, Dirichlet problem, the heat equation, the Schrodinger equation, conjugate functions, Hilbert transforms, ergodic theory, harmonic functions, singular integrals etc. All of them have the same pattern; we are interested in maximal operator. The key property to understand the Hardy-Littlewood maximal operator is the so called "weak type" estimate or property of M. Several mathematicians have worked on Hardy-Littlewood maximal functions. For example (Mingquan, 2016) proof that for $1 < p < \infty$, the L^p norm of the truncated centered Hardy-Littlewood maximal operator M_α^c equals the norm of the centered Hardy-Littlewood maximal operator for all $0 < \alpha < \infty$. (Martin-Reyes, 1993) gives simple proof of the characterization of the weights for which the one-sided Hardy-Littlewood maximal functions apply $L^p(W)$ into$L^p(W)$, where W is a nonnegative measurable function. We will like to extend the existing work in (Martin-Reyes, 1993) by looking at the generalization and modification of Hardy-Littlewood maximal functions.

Hardy-Littlewood Maximal Function

Defintion 1.1: Given $f \in L^1_{loc}(\mathbb{R})$, we define

$$Mf(x) = \sup_{x\in B}\frac{1}{|B|}\int_B |f(y)|dy \qquad (1)$$

where B is an integral containing x, and $|B|$ is the Lebesgue measure of B. The function $Mf(x)$ is called the maximal function of Hardy-Littlewood and the operator

$$M : f \mapsto Mf$$

It is called Hardy-Littlewood's maximal operator. M is not linear but sub-linear in the sense that

$$M(f+g) \le Mf + Mg, M(\alpha f) = |\alpha| Mf$$

Proof

$$Mf(x) = \sup_{x\in B}\frac{1}{|B|}\int_B |f(y)|dy$$

$$M(f+g)(x) = \sup_{x\in B}\frac{1}{|B|}\int_B |f(y)+g(y)|dy$$

$$\le \sup_{x\in B}\frac{1}{|B|}\int_B (|f(y)|+|g(y)|)dy$$

$$= \sup_{x\in B}\frac{1}{|B|}\int_B |f(y)|dy + \sup_{x\in B}\frac{1}{|B|}\int_B |g(y)|dy$$

$$= Mf(x) + Mg(x)$$

$$= (Mf + Mg)(x)$$

$$\therefore M(f+g) \le Mf + Mg \qquad (2)$$

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Also

$$M(\alpha f)(x) = \sup_{x\in B} \frac{1}{|B|}\int_B |\alpha f(y)|dy$$
$$= \sup_{x\in B} \frac{1}{|B|}\int_B |\alpha||f(y)|dy$$
$$= |\alpha| \sup_{x\in B} \frac{1}{|B|}\int_B |f(y)|dy$$
$$= |\alpha| Mf(x)$$
$$\therefore M(\alpha f) = |\alpha| Mf \qquad (3)$$

Combining (2) and (3) the result follows immediately.

Definition 1.2: The one-sided maximal functions M^+f and M^-f of a function $f \in L^1_{loc}(\mathbb{R})$ is given by (Sawyer, 1986) as

$$M^+f = \sup_{h>0} \frac{1}{h}\int_x^{x+h} |f|$$
$$M^-f = \sup_{h>0} \frac{1}{h}\int_{x-h}^{x} |f|$$

The operators $M^+f(x)$ and $M^-f(x)$ are interesting because they control some one-sided operators such as singular integrals with kernels supported in $(-\infty, 0)$ or $(0, \infty)$.

The results of the operators M_g^+ and M_g^- are defined as

$$M_g^+f(x) = \sup_{h>0} \frac{\int_x^{x+h} |f|g}{\int_x^{x+h} g}$$
$$M_g^-f(x) = \sup_{h>0} \frac{\int_{x-h}^{x} |f|g}{\int_{x-h}^{x} g}$$

Where g is a positive locally integrable function.

Lemma 1.1: The function Mf is lower-semicontinuous, hence measurable.

Proof

Let $M(x_0) > \beta$. Then there is a ball B containing x such that

$$\frac{1}{|B|}\int_B |f(y)|dy > \beta$$

Then $Mf(x) > \beta$ for every $x \in B$.

Definition 1.3: The classsical Hardy-Littlewood maximal function is given by

$$M'f(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy$$

i.e limited to the averages of $|f|$ over balls centered at x.

clearly, $M'f(x) \le Mf(x)$

$M'f$ is not necessarily lower-semicontinuous. The measurability of $M'f$ follows from the fact that the map

$$F(x,r) = \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy \qquad (4)$$

is continuous in r, so that the sup in (4) can be limited to $r \in \mathbb{Q}$.

Definition 1.4: The centered Hardy-Littlewood maximal function is given by

$$M^c f(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy \qquad (5)$$

and the uncentered Hardy-Littlewood maximal function is

$$Mf(x) = \sup_{x\in B} \frac{1}{|B|}\int_B |f(y)|dy$$

It is clear that

$$2^n M^c f(x) \ge Mf(x) \ge M^c f(x)$$

holds for all $x \in \mathbb{R}^n$. Both M and M^c are sublinear operators. It is very difficult to calculate the exact norm of M and M^c. The basic real-variable construct was introduced by (Hardy and Littlewood, 1930) for $n = 1$ and (Wiener, 1939) for $n \ge 2$.

Definition 1.5: Trucated Centered Hardy-Littewood maximal operator is given by

$$M_\alpha^c f(x) = \sup_{0<r<\alpha} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy \qquad (6)$$

and the Trucated Uncentered Hardy-Littlewood maximal operator

$$M_\alpha f(x) = \sup_{0<r<\alpha, |y-x|<r} \frac{1}{|B(y,r)|}\int_{B(y,r)} |f(t)|dt \qquad (7)$$

For $x \in \mathbb{R}^n$ and some real positive number α.

We can deduce from (5), (6) and (7) that

$$M^c f(x) \ge M_\beta^c f(x) \ge M_\alpha^c f(x) \qquad (8)$$

and

$$Mf(x) \ge M_\beta f(x) \ge M_\alpha f(x) \qquad (9)$$

$\forall\, x \in \mathbb{R}^n$, provided $\alpha \le \beta$.

It follows from (8) and (9), as the sublinear operators

$$||M^c||_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \ge ||M_\beta^c||_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \ge ||M_\alpha^c||_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \qquad (10)$$

and

$$||M||_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \geq ||M_\beta||_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \geq ||M_\alpha||_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \quad (11)$$

If $\alpha \leq \beta$, for $1 < p \leq \infty$.
This shows that for $1 < p \leq \infty$, the $L^p(\mathbb{R}^n)$ norm of the centered Hardy-Littlewood maximal operator is greater or equals to trucated centered Hardy-Littlewood maximal operator. Also, for $1 < p \leq \infty$, the L^p norm of the uncentered Hardy-Littlewood maximal operator is greater or equals to trucated uncentered Hardy-Littlewood maximal operator.

Relationship Between Hardy-Littewood Maximal Operators

Theorem 2.1: Let M_α^c be defined by (6) and $\alpha > 0$. Then

$$||M_\alpha^c||_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = ||M^c||_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}$$

That is, the norm of Trucated Centered Hardy-Littlewood maximal operator and the Trucated Uncentered Hardy-Littlehood maximal operator are exactly the same on a Lebsegue measure[6].

Theorem 2.2: Let M_α^c be defined by (6) and $\alpha > 0$. Then

$$||M_\alpha^c||_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)} = ||M^c||_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)}$$

Theorem 2.3: Let M_α be defined by (7) and $\alpha > 0$. Then

$$||M_\alpha\,||_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)} = ||M||_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)}$$

Theorem 2.4: Let M_α be defined by (7) and $\alpha > 0$. Then

$$||M_\alpha\,||_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = ||M||_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}$$

Lemma 2.1: Suppose that μ is a positive measure on a δ-algebra $\mathbb{M}$. If $A_1 \subset A_2 \subset A_3 \dots A_n \in \mathbb{M}$, and $A = \bigcup_{n=1}^{\infty} A_n$, then

$$\lim_{n\to\infty} \mu(A_n) = \mu(A)$$

Lemma 2.2: Suppose that the operators M^c and M_α^c are defined as in (8) and (10). The equality

$$dM^c f(\lambda) = \lim_{\alpha\to\infty} dM_\alpha^c f(\lambda)$$

Holds for all $f \in L^p(\mathbb{R}^n)$ and $\lambda > 0$.

Proof
For a fixed $x \in \mathbb{R}^n$, by the definition of M^c in (8), associate to each ε a ball $B(x, r_\varepsilon)$ which satisfies

$$\frac{1}{|B(x,r_\varepsilon)|}\int_{B(x,r_\varepsilon)} |f(y)|dy > M^c f(x) - \varepsilon$$

Now taking $\alpha > r_\varepsilon$, it follows from the definition of M_α^c that

$$M_\alpha^c f(x) \geq \frac{1}{|B(x,r_\varepsilon)|}\int_{B(x,r_\varepsilon)} |f(y)|dy > M^c f(x) - \varepsilon$$

Note that $M_\alpha^c f(x)$ increases as $\alpha \to \infty$. Thus we have

$$\lim_{\alpha\to\infty} M_\alpha^c \geq Mf \quad (12)$$

Clearly, we have

$$M_\alpha^c f \leq Mf \quad (13)$$

Combining (12) and (13), we have

$$\lim_{\alpha\to\infty} M_\alpha^c f = M^c f$$

Then

$$\lim_{n\to\infty} M_n^c f = M^c f$$

We set

$$A_n = \{x \in \mathbb{R}^n : M_n^c f(x) > \lambda\}$$

and

$$A = \{x \in \mathbb{R}^n : M^c f(x) > \lambda\}$$

We have $A_n \subset A_{n+1}$ for $n = 1,2, \dots$, and $A = \bigcup_{n=1}^{\infty} A_n$. It follows from Lemma 2.1 and the definition of the distribution function that

$$dM^c f(\lambda) = |A| = \lim_{n\to\infty} |A_n| = \lim_{n\to\infty} dM_n^c f(\lambda) = \lim_{\alpha\to\infty} dM_\alpha^c f(\lambda)$$

This is our desired result.

Generalization Of One-Sided Maximal Function
The natural generalization of M^+ in $\mathbb{R}^n$ is the following: given $x = (x_1, x_2, \dots x_n)$ we have

$$M^{+\cdots+} f(x) = \sup_{h>0} \frac{1}{h^n}\int_{Q_x(h)} |f(y)|dy$$

Where

$$Q_x(h) = [x_1, x_1 + h) \times [x_2, x_2 + h) \times \dots \times [x_n, x_n + h)$$

In $\mathbb{R}$ we have two one-sided operators. In $\mathbb{R}^n$ we obviously have 2^n one-sided operators that we do no write explicitly.

Given $x = (x_1, x_2, \dots x_n) \in \mathbb{R}^n$, let us assume

$$N^{+\cdots+}f(x) = \sup_{h>0} \frac{1}{h^n} \int_{Q_x^+(h)} |f(y)|dy$$

Where

$$Q_x^+(h) = [x_1 + h, x_1 + 2h) \times [x_2 + h, x_2 + 2h) \times \ldots \times [x_n + h, x_n + 2h)$$

For $n = 1$, M^+ and N^+ are equivalent.
It is clear that $N^{+\cdots+}f(x) \leq 2^n M^{+\cdots+}f(x)$ but there is no constant $C > 0$ satisfying

$$CN^{+\cdots+}f(x) \geq M^{+\cdots+}f(x)$$

for $n > 1$.

If $Q = [x_1, x_1 + h) \times [x_2, x_2 + h) \times \ldots \times [x_n, x_n + h)$ is a cube denote

$$Q^- = \left[x_1, x_1 + \frac{h}{2}\right) \times \left[x_2, x_2 + \frac{h}{2}\right) \times \ldots \times [x_n, x_n + \frac{h}{2})$$

and

$$Q^+ = \left[x_1 + \frac{h}{2}, x_1 + h\right) \times \left[x_2 + \frac{h}{2}, x_2 + h\right) \times \ldots \times [x_n + \frac{h}{2}, x_n + h)$$

Let $A_x^+ = \{Q dyadic : x \in Q^-\}$
The one-sided dyadic maximal function is defined by

$$M_d^{+\cdots+}f(x) = \sup_{Q \in A_x^+} \frac{1}{|Q^+|} \int_{Q^+}^{\cdot} |f(y)|dy$$

$$M_d f(x) = \sup_{Q\ dyadic: x \in Q} \frac{1}{|Q^+|} \int_{Q}^{\cdot} |f(y)|dy$$

M_d is the classical dyadic maximal operator.

Theorem 3.1: Let $f \in L^1(\mathrm{R})$. Then, for every $t > 0$, the set

$$E_t = \{x \in \mathrm{R} : Mf(x) > t\}$$

satisfies the following

$$|E_t| \leq \frac{2}{t} \int_{E_t} |f(x)|dx$$

Theorem 3.2: Let f measurable in R and $t > 0$. Then

$$|E_t| \leq \frac{4}{t} \int_{\{x: |f(x)| > \frac{t}{2}\}} |f(x)|dx$$

Proof

We can assume that $f \in L^1$. Define

$$f_1 = \begin{cases} f(x) & \text{if } |f(x)| > \frac{t}{2} \\ 0 & \text{otherwise} \end{cases}$$

Then, $f = f_1 + f_2$, with $|f_2| \leq \frac{t}{2}$ and $f_i \in L^1, i = 1,2$ we have $Mf_2 \leq \frac{t}{2}$

and

$$Mf(x) \leq Mf_1(x) + Mf_2(x) \leq Mf_1(x) + \frac{t}{2}$$

Then

$$E_t \subset \{x : Mf_1(x) > \frac{t}{2}\}$$

Now apply Theorem 3.1 to f_1 to obtain

$$\begin{aligned} |E_t| &\leq |\{x : Mf_1(x) > \frac{t}{2}\}| \\ &\leq \frac{4}{t} \int_{\mathrm{R}} |f_1(x)|dx \\ &= \frac{4}{t} \int_{\{x:|f(x)| > \frac{t}{2}\}} |f_1(x)|dx \\ &= \frac{4}{t} \int_{\{x:|f(x)| > \frac{t}{2}\}} |f(x)|dx \end{aligned}$$

Theorem 3.3: Given $1 < p < \infty$ there exist a constant C_p such that for every $f \in L^p$

$$\| Mf \|_{L^p} \leq C_p \| f \|_{L^p}$$

Proof

We will use distribution function, Theorem 3.1, 3.2 to obtain

$$\begin{aligned} \int_{\mathrm{R}} (Mf(x))^p dx &= p\int_0^\infty t^{p-1} |\{x : Mf(x) > t\}| dt \\ &\leq p\int_0^\infty t^{p-1} \frac{4}{t} \int_{\{x:|f(x)| > \frac{t}{2}\}} |f(x)| dxdt \\ &= 4p\int_0^\infty t^{p-1} \int_{\{x:|f(x)| > \frac{t}{2}\}} |f(x)| dxdt \\ &= 4p\int_0^\infty \int_0^{2|f(x)|} t^{p-2} |f(x)| dxdt \\ &= 4p \frac{2^{p-1}}{p-1} \int_{\mathrm{R}} |f(x)|^p dx \end{aligned}$$

Modified Hardy-Littlewood Maximal Function

Defintion 4.1: Let f be a nonnegative extended real-valued Lebesgue measurable function on $\mathbb{R}$ and λ be Lebesgue for $\mathbb{R}$. Then

$$M_r f(x) = \sup\left\{\frac{1}{\lambda(I)} \int_I^{\cdot} f d\lambda : I = [x, u], x < u < \infty\right\}$$

$$M_l f(x) = \sup\left\{\frac{1}{\lambda(I)} \int_I^{\cdot} f d\lambda : I = [u, x], -\infty < u < x\right\}$$

$Mf(x) = \sup\left\{\frac{1}{\lambda(I)} : \text{I is a closed interval containing x}\right\}$

$M_r f(x)$, $M_l f(x)$ and $Mf(x)$ are three diferent maximal averages of the functions f. All of the maximal theorems are inequalities giving bounds for the integral of one of the maximal functions composed with a monotonic function. The mapping M_r, M_l and M carry certain function spaces to others.

Theorem 4.1: $Mf = Max(M_r f,\ M_l f)$
See the proof in (Keith, 1965).

Definition 4.2: Let f be a locally integrable function on a metric measure space (X, μ). Then the k times modified centered Hardy-Littlewood maximal function $M_k f$ of f is defined as follows

$$M_k f = \sup_{r>0} \frac{1}{\mu(B(x, kr))} \int_{B(x,r)} |f(y)| d\mu(y)$$

We call the operator M_k the k times modified centered Hardy-Littlewood maximal operator. The k times modified uncentered Hardy-Littlewood maximal function $M_k f$ of f is defined as follows

$$M_{k,uc}\, f(x) = \sup_{x\in B(y,r)} \frac{1}{\mu\big(B(x, kr)\big)} \int_{B(x,r)} |f(z)| d\mu(z)$$

Clearly, the pointwise ineqiualities

$$M_k f \leq M_{k'} (k' \leq k)$$

and

$$M_{k,uc} f \leq M_{k',uc} (k' \leq k)$$

holds for any locally integrable function f on (X, μ). $M_{k,uc} f(x)$ is lower semicontinuous for any locally integrable function f.

Defintion 4.3: The n-dimensional maximal operator M is said to satisfy a weak type(1,1) inequality if there exists a constant c such that for every $f \in L^1(\mathbb{R}^n)$ and every $\alpha > 0$ we have

$$\alpha|\{Mf > \alpha\}| \leq c||f||_1$$

The corresponding extension to $\mathbb{R}^n$ of the Hardy-Littlewood maximal function is given by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x,r)} |f(y)| dy$$

As in the one dimensional case this definition corresponds to the centered maximal function and for the uncentered we require the basis simply to contain the point. Note that the uncentered is contained in a centered ball with double radius. More generally, one can start with a fixed set B containing the origin and define a maximal function using all the family of sets obtained using dilations and translations of B:

$$M_B f(x) = \sup_{r>0} \frac{1}{|rB|} \int_{rB} |f(x + y)| dy$$

If there are two balls centered at the origin with radii r_1 and r_2 such that $B(0, r_1) \subset B \subset B(0, r_2)$, then M_B is equivalent to M in the sense that the quotient $Mf(x)/M_B f(x)$ is bounded above and below by constant depending only on r_1, r_2 and the dimension, and not on f or x. In particular, this is true when B is the ball defined by an l^p −norm in $\mathbb{R}^n$.

Conclusion: In this paper, the research shows the centered Hardy-Littlewood maximal function and uncentered Hardy-Littlewood maximal funtion together with their k times modification and the generalization of one-sided maximal functions.

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